# On $q$-de Rham cohomology via $$-rings

# ON $q$-DE RHAM COHOMOLOGY VIA $\Lambda$-RINGS

J.P.PRIDHAM

ABSTRACT. We show that Aomoto's $q$-deformation of de Rham cohomology arises as a natural cohomology theory for $\Lambda$-rings. Moreover, Scholze's $(q-1)$-adic completion of $q$-de Rham cohomology depends only on the Adams operations at each residue characteristic. This gives a fully functorial cohomology theory, including a lift of the Cartier isomorphism, for smooth formal schemes in mixed characteristic equipped with a suitable lift of Frobenius. If we attach $p$-power roots of $q$, the resulting theory is independent even of these lifts of Frobenius, refining a comparison by Bhatt, Morrow and Scholze.

## INTRODUCTION

The $q$-de Rham cohomology of a polynomial ring is a $\mathbb{Z}[q]$-linear complex given by replacing the usual derivative with the Jackson $q$-derivative $\nabla_q(x^n) = [n]_q x^{n-1}dx$, where $[n]_q$ is Gauss' $q$-analogue $\frac{q^n-1}{q-1}$ of the integer $n$. In [Sch2], Scholze discussed the $(q-1)$-adic completion of this theory for smooth rings, explaining relations to $p$-adic Hodge theory and singular cohomology, and conjecturing that it is independent of co-ordinates.

We show that $q$-de Rham cohomology naturally arises as a functorial invariant of $\Lambda$-rings (Theorem 1.11), and that its $(q-1)$-adic completion depends only on a $\Lambda_P$-ring structure (Theorem 2.7), for $P$ the set of residue characteristics; a $\Lambda_P$ ring has a lift of Frobenius for each $p \in P$. This recovers the known equivalence between de Rham cohomology and complete $q$-de Rham cohomology over the rationals, while giving no really new functoriality statements for smooth schemes over $\mathbb{Z}$. However, in mixed characteristic, it means that complete $q$-de Rham cohomology depends only on a lift $\Psi^p$ of absolute Frobenius locally generated by co-ordinates with $\Psi^p(x_i) = x_i^p$. Given such data, we construct (Proposition 2.8) a quasi-isomorphism between Hodge cohomology and $q$-de Rham cohomology modulo $[p]_q$, extending the local lift of the Cartier isomorphism in [Sch2, Proposition 3.4].

Taking the Frobenius stabilisation of the complete $q$-de Rham complex of $A$ yields a complex resembling the de Rham–Witt complex. We show (Theorem 3.10) that up to $(q^{1/p^\infty} - 1)$-torsion, the $p$-adic completion of this complex depends only on the $p$-adic completion of $A[\zeta_{p^\infty}]$ (where $\zeta_n$ denotes a primitive $n$th root of unity), with no requirement for a lift of Frobenius or a choice of co-ordinates. The main idea is to show that the stabilised $q$-de Rham complex is in a sense given by applying Fontaine's period ring construction $A_{\mathrm{inf}}$ to the best possible perfectoid approximation to $A[\zeta_{p^\infty}]$. As a consequence, this shows (Corollary 3.11) that after attaching all $p$-power roots of $q$, $q$-de Rham cohomology in mixed characteristic is independent of choices, which was already known after base change to a period ring, via the comparisons of [BMS] between $q$-de Rham cohomology their theory $A\Omega^\bullet$.

We expect that the dependence of these cohomology theories either on Adams operations at the residue characteristics (for de Rham) or on $p$-power roots of $q$ (for variants of de Rham–Witt) is unavoidable, and that the conjectures of [Sch2] might thus be slightly optimistic. Some of the strongest evidence for the conjectures is provided by the lifts of the Cartier isomorphisms, which rely on a choice of Frobenius. On the other hand, the comparison theorems of [BMS] can be seen as a manifestation of $q$-de Rham–Witt complexes; although they do not require a lift of Frobenius, they involve all $p$-power roots of $q$.

The essence of our construction of $q$-de Rham cohomology of $A$ over $R$ is to set $q$ to be an element of rank 1 for the $\Lambda$-ring structure, and to look at flat $\Lambda$-rings $B$ over $R[q]$ equipped with morphisms $A \to B/(q-1)$ of $\Lambda$-rings over $R$. If these seem unfamiliar, reassurance should be provided by the observation that $(q-1)B$ carries $q$-analogues of divided power operations (Remark 1.4).

For the variants of de Rham–Witt cohomology in §3, the key to giving a characterisation independent of lifts of Frobenius is the factorisation of the tilting equivalence for perfectoid algebra via a category of $\Lambda_p$-rings.

I would like to thank Peter Scholze for many helpful comments, in particular about the possibility of a $q$-analogue of de Rham–Witt cohomology, and Michel Gros for spotting a missing hypothesis.

Contents



## 1. Comparisons for $\Lambda$-rings

We will follow standard notational conventions for $\Lambda$-rings. These are commutative rings equipped with operations $\lambda^i$ resembling alternating powers, in particular satisfying $\lambda^k(a+b) = \sum_{i=0}^k \lambda^i(a)\lambda^{k-i}(b)$, with $\lambda^0(a) = 1$ and $\lambda^1(a) = a$. For background, see [Bor] and references therein. The $\Lambda$-rings we encounter are all torsion-free, in which case the $\Lambda$-ring structure is equivalent to giving ring endomorphisms $\Psi^n$ for $n \in \mathbb{Z}_{>0}$ with $\Psi^{mn} = \Psi^m \circ \Psi^n$ and $\Psi^p(x) \equiv x^p \mod p$ for all primes $p$. If we write $\lambda_t(f) := \sum_{i\geq 0} \lambda^i(f)t^i$ and $\Psi_t(f) := \sum_{n\geq 1} \Psi^n(f)t^n$, then the families of operations are related by the formula $\Psi_t = -t\frac{d \log \lambda_{-t}}{dt}$.

We refer to elements $x$ with $\lambda^i(x) = 0$ for all $i > 1$ (or equivalently $\Psi^n(x) = x^n$ for all $n$) as elements of rank 1.

### 1.1. The $\Lambda$-ring $\mathbb{Z}[q]$.

**Definition 1.1.** Define $\mathbb{Z}[q]$ to be the $\Lambda$-ring with operations determined by setting $q$ to be of rank 1.

We now consider the $q$-analogues $[n]_q := \frac{q^n-1}{q-1} \in \mathbb{Z}[q]$ of the integers, with $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$, and $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$.

*Remark* 1.2. To see the importance of regarding $\mathbb{Z}[q]$ as a $\Lambda$-ring observe that the binomial expressions

$$\lambda^k(n) = \binom{n}{k}, \quad \lambda^k(-n) = (-1)^k\binom{n+k-1}{k}$$

have as $q$-analogues the Gaussian binomial theorems

$$\lambda^k([n]_q) = q^{k(k-1)/2}\binom{n}{k}_q, \quad \lambda^k(-[n]_q) = (-1)^k\binom{n+k-1}{k}_q,$$

as well as Adams operations

$$\Psi^i([n]_q) = [n]_{q^i}.$$

For any torsion-free $\Lambda$-ring, localisation at a set of elements closed under the Adams operations always yields another $\Lambda$-ring, since $\Psi^p(a^{-1}) - a^p = (\Psi^p(a)a^p)^{-1}(a^p - \Psi^p(a))$ is divisible by $p$.

**Lemma 1.3.** *For the $\Lambda$-ring structure on $\mathbb{Z}[x,y]$ with $x, y$ of rank 1, the elements*

$$\lambda^n(\tfrac{y-x}{q-1}) \in \mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}, x, y]$$

*are given by*

$$\lambda^k(\tfrac{y-x}{q-1}) = \frac{(y-x)(y-qx)\ldots(y-q^{k-1}x)}{(q-1)^k[k]_q!},$$
$$= \sum_{j=0}^{k} \frac{q^{j(j-1)/2}(-x)^j y^{k-j}}{[j]_q![k-j]_q!}.$$

*Proof.* The second expression comes from multiplying out the Gaussian binomial expansions. The easiest way to prove the first is to observe that $\lambda^k(\frac{y-x}{q-1})$ must be a homogeneous polynomial of degree $k$ in $x, y$, with coefficients in the integral domain $\mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}]$, and to note that

$$\lambda^k(\tfrac{q^n x - x}{q-1}) = \lambda^k([n]_q x) = q^{k(k-1)/2}\binom{n}{k}_q x^k.$$

Thus $\lambda^k(\frac{y-x}{q-1})$ agrees with the homogeneous polynomial above for infinitely many values of $\frac{y}{x}$, so must be equal to it. $\square$

*Remark* 1.4. Note that as $q \to 1$, Lemma 1.3 gives $(q-1)^k\lambda^k(\frac{y-x}{q-1}) \to \frac{(x-y)^k}{k!}$. Indeed, for any rank 1 element $x$ in a $\Lambda$-ring we have $\lambda_{(q-1)t}(\frac{x}{q-1}) = e_q(xt)$, the $q$-exponential, with multiplicativity and universality then implying that $\lambda_{(q-1)t}(\frac{a}{q-1})$ is a $q$-deformation of $\exp(at)$ for all $a$. Thus $(q-1)^k\lambda^k(\frac{a}{q-1})$ is a $q$-analogue of the $k$th divided power $(a^k/k!)$. An explicit expression comes recursively from the formula

$$[k]_q(q-1)\lambda^k(\tfrac{a}{q-1}) = \sum_{i>0} \lambda^i(a)\lambda^{k-i}(\tfrac{a}{q-1}).$$

**Lemma 1.5.** *For elements $x, y$ of rank 1, the $\Lambda$-subring of $\mathbb{Z}[q, \{(q^n-1)^{-1}\}_{n\geq 1}, x, y]$ generated by $q, x, y, \frac{y-x}{q-1}$ has basis $\lambda^k(\frac{y-x}{q-1})$ as a $\mathbb{Z}[q, x]$-module.*

*Proof.* The $\Lambda$-subring clearly contains the $\mathbb{Z}[q, x]$-module $M$ generated by the elements $\lambda^k(\frac{y-x}{q-1})$, which are also clearly linearly independent. Since $\mathbb{Z}[x, q]$ is a $\Lambda$-ring, it suffices to show that $M$ is closed under multiplication.

By Lemma 1.3, we know that

$$\lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-q^ix}{q-1}) = \tbinom{i+j}{i}_q \lambda^{i+j}(\tfrac{y-x}{q-1}).$$

We can rewrite $\frac{y-q^ix}{q-1} = \frac{y-x}{q-1} - [i]_q x$, so $\lambda^j(\frac{y-q^ix}{q-1}) - \lambda^j(\frac{y-x}{q-1})$ lies in the $\mathbb{Z}[q, x]$-module spanned by $\lambda^m(\frac{y-x}{q-1})$ for $m < j$. By induction on $j$, it thus follows that

$$\lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-q^ix}{q-1}) - \lambda^i(\tfrac{y-x}{q-1})\lambda^j(\tfrac{y-x}{q-1}) \in M,$$

so the binomial expression above implies $\lambda^i(\frac{y-x}{q-1})\lambda^j(\frac{y-x}{q-1}) \in M$. □

1.2. **$q$-cohomology of $\Lambda$-rings.**

**Definition 1.6.** Given a $\Lambda$-ring $R$, say that $A$ is a $\Lambda$-ring over $R$ if it is a $\Lambda$-ring equipped with a morphism $R \to A$ of $\Lambda$-rings. We say that $A$ is a flat $\Lambda$-ring over $R$ if $A$ is flat as a module over the commutative ring underlying $R$.

**Definition 1.7.** Given a morphism $R \to A$ of $\Lambda$-rings, we define the category $\mathrm{Strat}^q_{A/R}$ to consist of flat $\Lambda$-rings $B$ over $R[q]$ equipped with a compatible morphism $A \to B/(q-1)$, such that the map $A \to B/(q-1)$ admits a lift to $B$; a choice of lift is not taken to be part of the data, so need not be preserved by morphisms.

More concisely, $\mathrm{Strat}^q_{A/R}$ is the Grothendieck construction of the functor

$$(\mathrm{Spec}\, A)^q_{\mathrm{strat}} \colon\ B \mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda,R}(A, B) \to \mathrm{Hom}_{\Lambda,R}(A, B/(q-1)))$$

on the category $f\Lambda(R[q])$ of flat $\Lambda$-rings over $R[q]$.

**Definition 1.8.** Given a flat morphism $R \to A$ of $\Lambda$-rings, define $\mathrm{qDR}(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit of the functor

$$\begin{aligned} \mathrm{Strat}^q_{A/R} &\to \mathrm{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

Equivalently, can we follow the approach of [Gro, Sim] towards the de Rham stack by regarding $\mathrm{qDR}(A/R)$ as the quasi-coherent cohomology complex of $(\mathrm{Spec}\, A)^q_{\mathrm{strat}}$. Writing $\varnothing\colon\ f\Lambda(R[q]) \to \mathrm{Mod}_{R[q]}$ for the forgetful functor to the category of $R[q]$-modules, and $[f\Lambda(R[q]), \mathrm{Set}]$ for the category of set-valued functors on $f\Lambda(R[q])$, we have

$$\mathrm{qDR}(A/R) = \mathbf{R}\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\mathrm{Spec}\, A)^q_{\mathrm{strat}}, \varnothing),$$

coming from the right-derived functor of the functor $\mathrm{Hom}_{[f\Lambda(R[q]),\mathrm{Set}]}((\mathrm{Spec}\, A)^q_{\mathrm{strat}}, -)$ of natural transformations with source $(\mathrm{Spec}\, A)^q_{\mathrm{strat}}$.

**Definition 1.9.** Given a polynomial ring $R[x]$, recall from [Sch2] that the $q$-de Rham (or Aomoto–Jackson) cohomology $q\text{-}\Omega^\bullet_{R[x]/R}$ is given by the complex

$$R[x][q] \xrightarrow{\nabla_q} R[x][q]dx, \quad \text{where} \quad \nabla_q(f) = \frac{f(qx) - f(x)}{x(q-1)}dx,$$

so $\nabla_q(x^n) = [n]_q x^{n-1}dx$.

Given a polynomial ring $R[x_1, \ldots, x_d]$, the $q$-de Rham complex $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_d]/R}$ is then set to be

$$q\text{-}\Omega^\bullet_{R[x_1]/R} \otimes_{R[q]} q\text{-}\Omega^\bullet_{R[x_2]/R} \otimes_{R[q]} \cdots \otimes_{R[q]} q\text{-}\Omega^\bullet_{R[x_d]/R},$$

so takes the form

$$R[x_1, \ldots, x_d][q] \xrightarrow{\nabla_q} \Omega^1_{R[x_1,\ldots,x_d]/R}[q] \xrightarrow{\nabla_q} \ldots \xrightarrow{\nabla_q} \Omega^d_{R[x_1,\ldots,x_d]/R}[q].$$

**Proposition 1.10.** *If $R$ is a $\Lambda$-ring and $x$ of rank 1, then $\mathrm{qDR}(R[x]/R)$ can be calculated by a cosimplicial module $U^\bullet$ given by setting $U^n$ to be the $\Lambda$-subring*

$$U^n \subset R[q, \{(q^m-1)^{-1}\}_{m\geq 1}, x_0, \ldots, x_n]$$

*generated by $q$ and the elements $x_i$ and $\frac{x_i - x_j}{q-1}$.*

*Proof.* For $X = \operatorname{Spec} R[x]$, the set-valued functor $X^q_{\mathrm{strat}}$ is not representable, but it can be resolved by the simplicial functor $\tilde{X}^q_{\mathrm{strat}}$ given by taking the Čech nerve of $\mathrm{Hom}_{\Lambda,R}(A, B) \to \mathrm{Hom}_{\Lambda,R}(A, B/(q-1))$, so

$$\begin{aligned}(\tilde{X}^q_{\mathrm{strat}})_n(B) &:= \overbrace{\mathrm{Hom}_{\Lambda,R}(A, B) \times_{\mathrm{Hom}_{\Lambda,R}(A,B/(q-1))} \cdots \times_{\mathrm{Hom}_{\Lambda,R}(A,B/(q-1))} \mathrm{Hom}_{\Lambda,R}(A, B)}^{n+1} \\ &= \mathrm{Hom}_{\Lambda,R}(A, \overbrace{B \times_{B/(q-1)} \cdots \times_{B/(q-1)} B}^{n+1}).\end{aligned}$$

Observe that any element of $(X^q_{\mathrm{strat}})_n(B)$ gives rise to a morphism $f\colon R[q, x_0, \ldots, x_n] \to B$ of $\Lambda$-rings over $R[q]$, with the image of $x_i - x_j$ divisible by $(q-1)$. Flatness of $B$ then gives a unique element $f(x_i - x_j)/(q-1) \in B$, so we have a map $f$ to $B$ from the free $\Lambda$-ring $L$ over $R[q, x_0, \ldots, x_n]$ generated by elements $z_{ij}$ with $(q-1)z_{ij} = x_i - x_j$.

Since $B$ is flat, it embeds in $B[\{(q^m-1)^{-1}\}_{m\geq 1}]$ (the only hypothesis we really need) implying that the image of $f$ factors through the image $U^n$ of $L$ in $R[q, \{(q^m - 1)^{-1}\}_{m\geq 1}, x_0, \ldots, x_n]$. To see that $(X^q_{\mathrm{strat}})_n$ is represented by $U^n$, we only now need to check that $U^n$ is itself flat over $R[q]$, which follows because the argument of Lemma 1.5 gives a basis

$$x_0^{r_0}\lambda^{r_1}(\tfrac{x_1-x_0}{q-1}) \cdots \lambda^{r_n}(\tfrac{x_n-x_{n-1}}{q-1})$$

for $U^n$ over $R[q]$. We therefore have $\mathrm{qDR}(R[x]/R) \simeq U^\bullet$. □

**Theorem 1.11.** *If $R$ is a $\Lambda$-ring, and the polynomial ring $R[x_1, \ldots, x_d]$ is given the $\Lambda$-ring structure for which the elements $x_i$ are of rank 1, then there are $R[q]$-linear zigzags of quasi-isomorphisms*

$$\begin{aligned}\mathrm{qDR}(R[x_1, \ldots, x_n]/R) &\simeq (\Omega^*_{R[x_1,\ldots,x_n]/R}[q], (q-1)\nabla_q) \\ \mathbf{L}\eta_{(q-1)}\mathrm{qDR}(R[x_1, \ldots, x_n]/R) &\simeq q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_n]/R},\end{aligned}$$

*where $\mathbf{L}\eta_{(q-1)}$ denotes derived décalage with respect to the $(q-1)$-adic filtration.*

*Proof.* It suffices to prove the first statement, the second following immediately by décalage. Since $(\operatorname{Spec} A \otimes_R A')^q_{\mathrm{strat}}(B) = (\operatorname{Spec} A)^q_{\mathrm{strat}}(B) \times (\operatorname{Spec} A')^q_{\mathrm{strat}}(B)$, and coproduct of flat $\Lambda$-rings over $R[q]$ is given by $\otimes_{R[q]}$, we have $\mathrm{qDR}((A \otimes_R A')/R) \simeq \mathrm{qDR}(A/R) \otimes^{\mathbf{L}}_{R[q]} \mathrm{qDR}(A'/R)$, so we may reduce to the case $A = R[x]$.

Proposition 1.10 gives $\mathrm{qDR}(R[x]/R) \simeq U^\bullet$, and in order to compare this with $q$-de Rham cohomology, we now consider the cochain complexes $\tilde{\Omega}^\bullet(U^n)$ given by

$$U^n \xrightarrow{(q-1)\nabla_q} \bigoplus_i U^n dx_i \xrightarrow{(q-1)\nabla_q} \bigoplus_{i<j} U^n dx_i \wedge dx_j \xrightarrow{(q-1)\nabla_q} \ldots.$$

In order to see that this differential is well-defined, observe that

$$\begin{aligned}(q-1)\nabla_{q,y}\lambda^k(\tfrac{y-x}{q-1}) &= y^{-1}(\lambda^k(\tfrac{qy-x}{q-1}) - \lambda^k(\tfrac{y-x}{q-1}))dy \\ &= y^{-1}(\lambda^k(y+\tfrac{y-x}{q-1}) - \lambda^k(\tfrac{y-x}{q-1}))dy \\ &= \lambda^{k-1}(\tfrac{y-x}{q-1})dy,\end{aligned}$$

and similarly

$$(q-1)\nabla_{q,x}\lambda^k(\tfrac{y-x}{q-1}) = \sum_{i\geq 1}(-1)^i x^{i-1}\lambda^{k-i}(\tfrac{y-x}{q-1})dx.$$

The first calculation also shows that the inclusion $\tilde{\Omega}^\bullet(U^{n-1}) \hookrightarrow \tilde{\Omega}^\bullet(U^n)$ is a quasi-isomorphism, since

$$(q-1)\nabla_{q,x_n}(f(x_0,\ldots,x_{n-1})\lambda^k(\tfrac{x_n-x_{n-1}}{q-1})) = f(x_0,\ldots,x_{n-1})\lambda^{k-1}(\tfrac{x_n-x_{n-1}}{q-1})dx_n.$$

By induction on $n$ we deduce that the inclusion $\tilde{\Omega}^\bullet(U^0) \hookrightarrow \tilde{\Omega}^\bullet(U^n)$, and hence the retraction of it given by the diagonal, is a quasi-isomorphism.

Now the complexes $\tilde{\Omega}^i(U^\bullet)$ are all acyclic for $i>0$, consisting of cosimplicial tensor products of $U^\bullet$ with cosimplicial symmetric powers of the acyclic complex given by $\mathbb{Z}dx_0 \oplus \cdots \oplus \mathbb{Z}dx_n$ in level $n$. We therefore have quasi-isomorphisms

$$U^\bullet \leftarrow \mathrm{Tot}\,\tilde{\Omega}^\bullet(U^\bullet) \to \tilde{\Omega}^\bullet(U^0)$$

of flat cochain complexes over $R[q]$, so

$$\mathrm{qDR}(R[x]/R) \simeq \tilde{\Omega}^\bullet(R[x]),$$

and we just observe that $\eta_{(q-1)}\tilde{\Omega}^\bullet(R[x]) = (\Omega^*_{R[x]/R}[q], (q-1)\nabla_q)$. □

*Remark* 1.12. Note that Theorem 1.11 implies that $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_n]/R}$ naturally underlies the décalage of a cosimplicial $\Lambda$-ring over $R[q]$. Even the underlying cosimplicial commutative ring structure carries more information than an $E_\infty$-structure when $\mathbb{Q} \not\subseteq R$.

*Remark* 1.13. The complex $(\Omega^*_{R[x_1,\ldots,x_n]/R}[q], (q-1)\nabla_q)$ is a more fundamental object than its décalage $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_n]/R}$, since it has a vestigial memory of the Hodge filtration.

There might be a natural formulation of the theorem not involving décalage, in terms of a $q$-analogue of the crystalline site for a $\Lambda$-ring $A$ over $R$, regarded as an $R[q]$-algebra via $R = R[q]/(q-1)$. Following Remark 1.4, this would involve extensions $B \to A$ of $\Lambda$-rings over $R[q]$ equipped with $q$-analogues of divided power operations on the augmentation ideals $I$, looking like $x \mapsto (q-1)^k\lambda^k(\frac{x}{q-1})$.

### 1.3. **Completed $q$-cohomology.**

**Definition 1.14.** Given a morphism $R \to A$ of $\Lambda$-rings, we define the category $\hat{\mathrm{Strat}}^q_{A/R} \subset \mathrm{Strat}^q_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.

**Definition 1.15.** Given a flat morphism $R \to A$ of $\Lambda$-rings, define $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R)$ to be the cochain complex of $R[\![q-1]\!]$-modules given by taking the homotopy limit of the functor

$$\begin{aligned} \hat{\mathrm{Strat}}^q_{A/R} &\to \mathrm{Ch}(R[\![q]\!]) \\ B &\mapsto B. \end{aligned}$$

The following is immediate:

**Lemma 1.16.** *Given a flat morphism $R \to A$ of $\Lambda$-rings, the complex $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R)$ is the derived $(q-1)$-adic completion of $\mathrm{qDR}(A/R)$.*

**Definition 1.17.** As in [Sch2, §3], given a formally étale map $\square\colon R[x_1,\ldots,x_d] \to A$, define $\widehat{q\text{-}\Omega}^\bullet_{A/R,\square}$ to be the complex

$$A[\![q-1]\!] \xrightarrow{\nabla_q} \Omega^1_{A/R}[\![q-1]\!] \xrightarrow{\nabla_q} \ldots \xrightarrow{\nabla_q} \Omega^d_{A/R}[\![q-1]\!],$$

where $\nabla_q$ is defined as follows. First note that the ring endomorphisms $\gamma_i$ of $R[x_1,\ldots,x_d][\![q-1]\!]$ given by $\gamma_i(x_j) = q^{\delta_{ij}}x_j$ extend uniquely to endomorphisms of $A[\![q-1]\!]$ which are the identity modulo $(q-1)$, then set

$$\nabla_q(f) := \sum_i \frac{\gamma_i(f)-f}{(q-1)x_i}dx_i.$$

Note that $\widehat{q\text{-}\Omega}^\bullet_{R[x_1,\ldots,x_d]/R}$ is just the $(q-1)$-adic completion of $q\text{-}\Omega^\bullet_{R[x_1,\ldots,x_d]/R}$.

**Proposition 1.18.** *If $R$ is a flat $\Lambda$-ring over $\mathbb{Z}$ and $\square\colon R[x_1,\ldots,x_d] \to A$ is a formally étale map of $\Lambda$-rings, the elements $x_i$ having rank $1$, then there are zigzags of $R[\![q]\!]$-linear quasi-isomorphisms*

$$\widehat{\mathrm{qDR}}(A/R) \simeq (\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q), \qquad \mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R) \simeq \widehat{q\text{-}\Omega}^\bullet_{A/R,\square}.$$

*The induced quasi-isomorphisms*

$$\widehat{\mathrm{qDR}}(A/R) \otimes^{\mathbf{L}}_{R[\![q-1]\!]} R \simeq (\Omega^*_{A/R}, 0), \qquad (\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}(A/R)) \otimes^{\mathbf{L}}_{R[\![q-1]\!]} R \simeq \Omega^\bullet_{A/R}$$

*are independent of the choice of framing.*

*Proof.* This is much the same as the proof of Theorem 1.11. The complex $\widehat{\mathrm{qDR}}(A/R)$ can be realised as a cosimplicial $\Lambda$-ring $U$, with $U^n$ the $(q-1)$-adically complete $\Lambda$-subring of $A^{\otimes_R(n+1)}[\![q-1]\!][\{(q^m-1)^{-1}\}_{m\geq 1}]$ generated by

$$A^{\otimes_R(n+1)}[\![q-1]\!] \quad \text{and} \quad (q-1)^{-1}\ker(A^{\otimes_R(n+1)} \to A)[\![q-1]\!].$$

Uniqueness of lifts with respect to the formally étale framing ensures that the endomorphisms $\gamma_i$ commute with the Adams operations, so are $\Lambda$-ring endomorphisms of $R$. Since the formal completion of $A \otimes_R A \to A$ is just the $\Lambda$-ring

$$A[\![(x_1-y_1),(x_2-y_2),\ldots,(x_d-y_d)]\!],$$

the calculations of Theorem 1.11 now adapt to give quasi-isomorphisms

$$(\Omega^*_{A/R}[\![q]\!], (q-1)\nabla_q) \leftarrow \mathrm{Tot}\,\tilde{\Omega}^\bullet(U^\bullet) \to U^\bullet,$$

where $\tilde{\Omega}^\bullet(U^n)$ is the $(q-1)$-adic completion of $(U^n \otimes_{A^{\otimes(n+1)}} (\Omega^*_{A/R})^{\otimes(n+1)}, (q-1)\nabla_q))$. Reduction of this or its décalage modulo $(q-1)$ replaces $\nabla_q$ with $d$, removing the dependence on co-ordinates. $\square$

*Remark* 1.19. As in [Sch2, Definition 7.3], there is a notion of $q$-connections on projective $A[\![q-1]\!]$-modules $M$. Adapting the ideas of Proposition 1.18, these will be equivalent to projective modules over $X^q_{\mathrm{strat}}$, so flat Cartesian $\widehat{\mathrm{qDR}}(A/R)$-modules $N$ with $N\otimes_{\widehat{\mathrm{qDR}}(A/R)} A[\![q-1]\!] = M$, together with a condition that the $(\widehat{\mathrm{qDR}}(A/R)/(q-1))$-module $N/(q-1)$ is just given by pullback of the $A$-module $M/(q-1)$.

Via Lemma 1.5, these data are equivalent to specifying an operator $\partial^1 \colon M \to \hat{\bigoplus}_{k_1,\ldots,k_d} M\lambda^{k_1}(\frac{y_1-x_1}{q-1})\cdots\lambda^{k_d}(\frac{y_d-x_d}{q-1})$ satisfying a cocycle condition and congruent to the identity modulo $(q-1)$. Such operators then arise from $q$-connections $(\nabla_{1,q},\ldots,\nabla_{d,q})$ as $q$-Taylor series

$$\partial^1(f) := \sum_{k_1,\ldots,k_d} (q-1)^{\sum k_i}(\nabla_{1,q}^{k_1}\ldots\nabla_{d,q}^{k_d})(f)\,\lambda^{k_1}(\tfrac{y_1-x_1}{q-1})\cdots\lambda^{k_d}(\tfrac{y_d-x_d}{q-1}).$$

## 2. Comparisons for $\Lambda_P$-rings

Since very few étale maps $R[x_1,\ldots,x_d]\to A$ give rise to $\Lambda$-ring structures on $A$, Proposition 1.18 is fairly limited in its scope for applications. We now show how the construction of $\mathrm{q\hat{D}R}$ and the comparison quasi-isomorphism survive when we weaken the $\Lambda$-ring structure by discarding Adams operations at invertible primes.

### 2.1. $q$-cohomology for $\Lambda_P$-rings.

Our earlier constructions for $\Lambda$-rings all carry over to $\Lambda_P$-rings, as follows.

**Definition 2.1.** Given a set $P$ of primes, we define a $\Lambda_P$-ring $A$ to be a $\Lambda_{\mathbb{Z},P}$-ring in the sense of [Bor]. This means that it is a coalgebra in commutative rings for the comonad given by the functor $W^{(P)}$ of $P$-typical Witt vectors. When a commutative ring $A$ is flat over $\mathbb{Z}$, giving a $\Lambda_P$-ring structure on $A$ is equivalent to giving commuting Adams operations $\Psi^p$ for all $p\in P$, with $\Psi^p(a)\equiv a^p \mod p$ for all $a$.

Thus when $P$ is the set of all primes, a $\Lambda_P$-ring is just a $\Lambda$-ring; a $\Lambda_\emptyset$-ring is just a commutative ring; for a single prime $p$, we write $\Lambda_p := \Lambda_{\{p\}}$, and note that a $\Lambda_p$-ring is a $\delta$-ring in the sense of [Joy].

**Definition 2.2.** Given a $\Lambda_P$-ring $R$, say that $A$ is a $\Lambda_P$-ring over $R$ if it is a $\Lambda_P$-ring equipped with a morphism $R\to A$ of $\Lambda_P$-rings. We say that $A$ is a flat $\Lambda_P$-ring over $R$ if $A$ is flat as a module over the commutative ring underlying $R$.

**Definition 2.3.** Given a morphism $R\to A$ of $\Lambda_P$-rings, we define the category $\mathrm{Strat}^{q,P}_{A/R}$ to consist of flat $\Lambda_P$-rings $B$ over $R[q]$ equipped with a compatible morphism $A\to B/(q-1)$, such that the map $A\to B/(q-1)$ admits a lift to $B$. We define the category $\hat{\mathrm{Strat}}^{q,P}_{A/R}\subset \mathrm{Strat}^q_{A/R}$ to consist of those objects which are $(q-1)$-adically complete.

More concisely, $\mathrm{Strat}^{q,P}_{A/R}$ (resp. $\hat{\mathrm{Strat}}^{q,P}_{A/R}$) is the Grothendieck construction of the functor

$$(\mathrm{Spec}\,A)^{q,P}_{\mathrm{strat}}\colon\ B\mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda_P,R}(A,B)\to \mathrm{Hom}_{\Lambda_P,R}(A,B/(q-1)))$$

of the category of flat $\Lambda_P$-rings (resp. $(q-1)$-adically complete flat $\Lambda_P$-rings) over $R[q]$.

**Definition 2.4.** Given a flat morphism $R \to A$ of $\Lambda_P$-rings, define $\mathrm{qDR}_P(A/R)$ to be the cochain complex of $R[q]$-modules given by taking the homotopy limit of the functor

$$\begin{aligned} \mathrm{Strat}^{q,P}_{A/R} &\to \mathrm{Ch}(R[q]) \\ B &\mapsto B. \end{aligned}$$

Define $\mathrm{q\hat{D}R}_P(A/R)$ to be the cochain complex of $R[\![q-1]\!]$-modules given by the corresponding homotopy limit over $\hat{\mathrm{Strat}}^{q,P}_{A/R}$.

Thus when $P$ is the set of all primes, we have $\mathrm{qDR}_P(A/R) = \mathrm{qDR}(A/R)$. At the other extreme, for $A$ smooth, $\mathrm{q\hat{D}R}_\emptyset(A/R)$ is the Rees construction of the Hodge filtration on the infinitesimal cohomology complex of $A$ over $R$ , with formal variable $(q-1)$. In more detail, there is a decreasing filtration $F$ of $\mathcal{O}_{\mathrm{inf}}$ given by powers of the augmentation ideal $\mathcal{O}_{\mathrm{inf}} \to \mathcal{O}_{\mathrm{Zar}}$, and $\mathrm{q\hat{D}R}_\emptyset(A/R) \simeq \bigoplus_{\nu \in \mathbb{Z}}(q-1)^{-\nu}\mathbf{R}\Gamma(\mathrm{Spec}\, A, F^\nu \mathcal{O}_{\mathrm{inf}})(q-1)^{-\nu}$.

**Lemma 2.5.** *For a set $P$ of primes, the forgetful functor from $\Lambda$-rings to $\Lambda_P$-rings has a right adjoint $W^{(\notin P)}$. There is a canonical ghost component morphism*

$$W^{(\notin P)}(B) \to \prod_{\substack{n \in \mathbb{N}:\\ (n,p)=1 \ \forall p \in P}} B,$$

*which is an isomorphism when $P$ contains all the residue characteristics of $B$.*

*Proof.* Existence of a right adjoint follows from the comonadic definitions of $\Lambda$-rings and $\Lambda_P$-rings. The ghost component morphism is given by taking the Adams operations $\Psi^n$ given by the $\Lambda$-ring structure on $W^{(\notin P)}(B)$, followed by projection to $B$. When $P$ contains all the residue characteristics of $B$, a $\Lambda$-ring structure is the same as a $\Lambda_P$-ring structure with compatible commuting Adams operations for all primes not in $P$, leading to the description above. □

Note that the big Witt vector functor $W$ on commutative rings thus factorises as $W = W^{(\notin P)} \circ W^{(P)}$, for $W^{(P)}$ the $P$-typical Witt vectors.

**Proposition 2.6.** *Given a morphism $R \to A$ of $\Lambda$-rings, and a set $P$ of primes, there are natural maps*

$$\mathrm{qDR}_P(A/R) \to \mathrm{qDR}(A/R), \qquad \mathrm{q\hat{D}R}_P(A/R) \to \mathrm{q\hat{D}R}(A/R),$$

*and the latter map is a quasi-isomorphism when $P$ contains all the residue characteristics of $A$.*

*Proof.* We have functors

$$\begin{aligned} (\mathrm{Spec}\, A)^q_{\mathrm{strat}} \circ W^{(\notin P)} &\colon\ B \mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda,R}(A, W^{(\notin P)}B) \to \mathrm{Hom}_{\Lambda,R}(A, (W^{(\notin P)}B)/(q-1))) \\ (\mathrm{Spec}\, A)^{q,P}_{\mathrm{strat}} &\colon\ B \mapsto \mathrm{Im}\,(\mathrm{Hom}_{\Lambda_P,R}(A, B) \to \mathrm{Hom}_{\Lambda_P,R}(A, B/(q-1))) \end{aligned}$$

on the category of flat $\Lambda_P$-rings over $R[q]$. There is an obvious map

$$(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1)),$$

and hence a natural transformation $(\mathrm{Spec}\, A)^q_{\mathrm{strat}} \circ W^{(\notin P)} \to (\mathrm{Spec}\, A)^{q,P}_{\mathrm{strat}}$, which induces the morphism $\mathrm{qDR}_P(A/R) \to \mathrm{qDR}(A/R)$ on cohomology.

When $P$ contains all the residue characteristics of $A$, the map $(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1))$ is just

$$\prod_{\substack{n\in\mathbb{N}:\\ (n,p)=1\ \forall p\in P}} B/(q^n-1) \to \prod_{\substack{n\in\mathbb{N}:\\ (n,p)=1\ \forall p\in P}} B/(q-1),$$

since the morphism $R[q] \to W^{(\notin P)}B$ is given by Adams operations, with $\Psi^n(q-1) = q^n - 1$.

We have $(q^n-1) = (q-1)[n]_q$, and $[n]_q$ is a unit in $\mathbb{Z}[\frac{1}{n}][\![q-1]\!]$, hence a unit in $B$ when $n$ is coprime to the residue characteristics. Thus the map $(W^{(\notin P)}B)/(q-1) \to W^{(\notin P)}(B/(q-1))$ gives an isomorphism whenever $B$ is $(q-1)$-adically complete and admits a map from $A$, so the transformation $(\operatorname{Spec} A)^q_{\mathrm{strat}} \circ W^{(\notin P)} \to (\operatorname{Spec} A)^{q,P}_{\mathrm{strat}}$ is a natural isomorphism on the category of flat $(q-1)$-adically complete $\Lambda_P$-rings over $R[q]$, and so $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_P(A/R) \xrightarrow{\simeq} \mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/R)$. □

Over $\mathbb{Z}[\frac{1}{P}]$, every $\Lambda_P$-ring can be canonically made into a $\Lambda$-ring, by setting all the additional Adams operations to be the identity. However, this observation is of limited use in establishing functoriality of $q$-de Rham cohomology, because the resulting $\Lambda$-ring structure will not satisfy the conditions of Proposition 1.18. We now give a more general result which does allow for meaningful comparisons.

**Theorem 2.7.** *If $R$ is a flat $\Lambda_P$-ring over $\mathbb{Z}$ and $\square\colon R[x_1,\ldots,x_d] \to A$ is a formally étale map of $\Lambda_P$-rings, the elements $x_i$ having rank 1, then there is a zigzag of $R[\![q]\!]$-linear quasi-isomorphisms*

$$\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_P(A/R) \simeq \widehat{q\text{-}\Omega}^\bullet_{A/R,\square}$$

*whenever $P$ contains all the residue characteristics of $A$.*

*Proof.* The key observation to make is that formally étale maps have a unique lifting property with respect to nilpotent extensions of flat $\Lambda_P$-rings, because the Adams operations must also lift uniquely. In particular, this means that the operations $\gamma_i$ featuring in the definition of $q$-de Rham cohomology are necessarily endomorphisms of $A$ as a $\Lambda_P$-ring.

Similarly to Proposition 1.18, $\widehat{\mathrm{qDR}}_P(A/R)$ is calculated using a cosimplicial $\Lambda_P$-ring given in level $n$ by the $(q-1)$-adic completion $\hat{U}^\bullet_{P,A}$ of the $\Lambda_P$-ring over $R[q]$ generated by $A^{\otimes_R(n+1)}[q]$ and $(q-1)^{-1}\ker(A^{\otimes_R(n+1)} \to A)[q]$. The observation above shows that $\hat{U}^n_{P,A} \cong \hat{U}^n_{P,R[x_1,\ldots,x_d]}\hat{\otimes}_{R[x_1,\ldots,x_d]}A$, changing base along $\square$ applied to the first factor.

As in Proposition 2.6, $\hat{U}^\bullet_{P,R[x_1,\ldots,x_d]}$ is just the $(q-1)$-adic completion of the complex $U^\bullet$ from Proposition 1.10. Further application of the key observation above then allows us to adapt the constructions of Theorem 1.11, giving the desired quasi-isomorphisms. □

2.2. **Cartier isomorphisms in mixed characteristic.** The only setting in which Theorem 2.7 leads to results close to the conjectures of [Sch2] is when $R = W^{(p)}(k)$, the $p$-typical Witt vectors of a field of characteristic $p$, and $A = \varprojlim_n A_n$ is a formal deformation of a smooth $k$-algebra $A_0$. Then any formally étale morphism $W^{(p)}(k)[x_1,\ldots,x_d] \to A$ gives rise to a unique compatible lift $\Psi$ of absolute Frobenius on $A$ with $\Psi(x_i) = x_i^p$, so gives $A$ the structure of a topological $\Lambda_p$-ring. The framing still affects the choice of $\Lambda_p$-ring structure, but at least such a structure is guaranteed to exist, giving rise to a complex $\mathrm{qDR}_P(A/R)^{\wedge_p} := \varprojlim_n \mathrm{qDR}_p(A/R) \otimes_R R_n$ depending only on the choice of $\Psi$, where $R_n = W_n^{(p)}(k)$.

Our constructions now allow us to globalise the quasi-isomorphism

$$(\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\square})^{\wedge_p}/[p]_q \simeq (\Omega^*_{A/R})^{\wedge_p}[\![q-1]\!]/[p]_q$$

of [Sch2, Proposition 3.4], where $\Omega^*_{A/R}$ denotes the complex $A \xrightarrow{0} \Omega^1_{A/R} \xrightarrow{0} \Omega^2_{A/R} \xrightarrow{0} \ldots$.

**Proposition 2.8.** *Take a smooth formal scheme $\mathfrak{X}$ over $R = W^{(p)}(k)$ equipped with a lift $\Psi$ of Frobenius which étale locally admits co-ordinates $\{x_i\}_i$ as above with $\Psi(x_i) = x_i^p$. Then there is a global quasi-isomorphism*

$$C_q^{-1}\colon\ (\Omega^*_{\mathfrak{X}/R})^{\wedge_p}[\![q-1]\!]/[p]_q \to (\mathbf{L}\eta_{(q-1)}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(\mathscr{O}_{\mathfrak{X}}/R))^{\wedge_p}/[p]_q$$

*in the derived category of étale sheaves on $\mathfrak{X}$.*

*Proof.* Functoriality of the construction $\mathrm{qDR}_p$ for rings with Frobenius lifts gives us a sheaf $\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}$ on $\mathfrak{X}$. We then have maps

$$\Psi^p\colon\ \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p} \to \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}$$
$$\mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q-1) \to \mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q^p-1),$$

and thus, denoting good truncation by $\tau$,

$$(q-1)^i\Psi^p\colon\ \tau^{\leq i}(\mathrm{qDR}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p}/(q-1)) \to (\mathbf{L}\eta_{(q-1)}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(\mathscr{O}_{\mathfrak{X}}/R)^{\wedge_p})/[p]_q;$$

the left-hand side is quasi-isomorphic to $\bigoplus_{j\leq i}(\Omega^j_{\mathscr{O}_{\mathfrak{X}}/R})^{\wedge_p}[-j]$ by Proposition 1.18.

Extending the construction $R[q]$-linearly and restricting to top summands therefore gives us the global map $C_q^{-1}$. For a local choice of framing, the map $\Psi^p$ necessarily corresponds via Theorem 2.7 to the chain map $adx^I \mapsto \Psi^p(a)x^{I(p-1)}dx^I$ on the complex $(\Omega^*_{A/R}[\![q-1]\!], (q-1)\nabla_q)$. This gives equivalences

$$(q-1)^i\Psi^p \simeq \sum_{j\leq i}(q-1)^{i-j}(\tilde{C}^{-1})^j$$

for Scholze's locally defined lifts $(\tilde{C}^{-1})^j\colon\ (\Omega^j_{A/R})^{\wedge_p}[-j] \to (\widehat{q\text{-}\Omega}^{\bullet}_{A/R,\square})^{\wedge_p}/[p]_q$ of the Cartier quasi-isomorphism. The local calculation of [Sch2, Proposition 3.4] then ensures that $C_q^{-1}$ is a quasi-isomorphism. $\square$

## 3. Functoriality via analogues of de Rham–Witt cohomology

In order to obtain a cohomology theory for smooth commutative rings rather than for $\Lambda_P$-rings, we now consider $q$-analogues of de Rham–Witt cohomology. Our starting point is to observe that if we allow roots of $q$, we can extend the Jackson differential to fractional powers of $x$ by the formula

$$\nabla_q(x^{m/n}) = \frac{q^{m/n}-1}{q-1}x^{m/n}d\log x,$$

so terms such as $[n]_{q^{1/n}}x^{m/n}$ have integral derivative, where $[n]_{q^{1/n}} = \frac{q-1}{q^{1/n}-1}$.

### 3.1. Motivation.

**Definition 3.1.** Given a $\Lambda_P$-ring $B$, define $\Psi^{1/P^\infty}B$ to be the smallest $\Lambda_P$-ring containing $B$ on which the Adams operations are automorphisms.

In the case $P=\{p\}$, the $\Lambda_p$-ring $\Psi^{1/p^\infty}B$ is thus the colimit of the diagram

$$B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} B \xrightarrow{\Psi^p} \ldots.$$

The proof of Theorem 2.7 allows us to replace $\widehat{\mathrm{qDR}}_p(A/R)$ with the complex $(\Omega^*_{A/R}[\![q-1]\!],(q-1)\nabla_q)$; under this quasi-isomorphism, the Adams operations on $A$ extend to $\Omega^*_{A/R}[\![q-1]\!]$ by setting $\Psi^n(dx_i):=x_i^{n-1}dx_i$. As an immediate consequence we have:

**Lemma 3.2.** *If $R$ is a flat $\Lambda_p$-ring over $\mathbb{Z}$ with $\Psi^{1/p^\infty}R=R$ and residue characteristic $p$, then $\Psi^{1/p^\infty}\mathrm{qDR}_p(R[x]/R)\simeq(\Omega^*_{R[x^{1/p^\infty}]/R}[q^{1/p^\infty}],(q-1)\nabla_q)$, so the décalage $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(R[x]/R)$ is quasi-isomorphic to the $(q-1)$-adic completion of the complex*

$$\begin{aligned}&\{a\in R[x^{1/p^\infty},q^{1/p^\infty}] \;:\; \nabla_q a\in R[x^{1/p^\infty},q^{1/p^\infty}]d\log x\}\xrightarrow{\nabla_q}\\ &\{b\,d\log x\in R[x^{1/p^\infty},q^{1/p^\infty}]d\log x \;:\; b(0,q)=0\}.\end{aligned}$$

Thus in level 0 (resp. level 1), $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}(R[x]/R)$ is spanned by elements of the form $[p^n]_{q^{1/p^n}}x^{m/p^n}$ (resp. $x^{m/p^n}d\log x$), so setting $q^{1/p^\infty}=1$ gives a complex whose $p$-adic completion is the $p$-typical de Rham–Witt complex.

**Lemma 3.3.** *Let $R$ and $A$ be flat $p$-adically complete $\Lambda_p$-algebras over $\mathbb{Z}_p$, with $\Psi^{1/p^\infty}R=R$ and, for elements $x_i$ of rank 1, a map $\square\colon R[x_1,\ldots,x_d]^{\wedge_p}\to A$ of $\Lambda_p$-rings which is a flat $p$-adic deformation of an étale map. Then the map*

$$(R[q^{1/p^\infty}]\otimes_{R[q]}\mathbf{L}\eta_{(q-1)}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}\to\mathbf{L}\eta_{(q-1)}(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}$$

*is a quasi-isomorphism.*

*Proof.* The map $\Psi^p\colon A\otimes_{R[x_1,\ldots,x_d]}R[x_1^{1/p},\ldots,x_d^{1/p}]\to A$ becomes an isomorphism on $p$-adic completion, because $\square$ is flat and we have an isomorphism modulo $p$. Thus

$$\Psi^{1/p^\infty}A\cong A[x_1^{1/p^\infty},\ldots,x_d^{1/p^\infty}]^{\wedge_p}:=(A\otimes_{R[x_1,\ldots,x_d]}R[x_1^{1/p^\infty},\ldots,x_d^{1/p^\infty}])^{\wedge_p}$$

Combined with the calculation of Lemma 3.2, this gives us a quasi-isomorphism between $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/R))^{\wedge_p}$ and the $(p,q-1)$-adic completion of

$$\bigoplus_\alpha A[\![q-1]\!]x_1^{\alpha_1}\ldots x_d^{\alpha_d}dx^I,$$

where $\alpha\in p^{-\infty}\mathbb{Z}^d$ such that $0\le\alpha_i<1$ if $i\notin I$ and $-1<\alpha_i\le 0$ if $i\in I$.

We then observe that the contributions to the décalage $\eta_{(q-1)}$ from terms with $\alpha\neq 0$ must be acyclic, via a contracting homotopy defined by the restriction to $\eta_{(q-1)}$ of the $q$-integration map

$$fx_1^{\alpha_1}\ldots x_d^{\alpha_d}dx^I\mapsto fx_1^{\alpha_1}\ldots x_d^{\alpha_d}\sum_{i\in I}\pm x_i[\alpha_i]_q^{-1}dx^{(I\setminus i)},$$

where $[\frac{m}{p^n}]_q^{-1}=[m]_{q^{1/p^n}}^{-1}[p^n]_{q^{1/p^n}}$ for $m$ coprime to $p$, noting that $[m]_{q^{1/p^n}}$ is a unit in $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$. $\square$

*Remark* 3.4. The endomorphism given on $\Psi^{1/P^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_P(A/R)$ by
$$a \mapsto \Psi^{1/n}([n]_q a) = [n]_{q^{1/n}}\Psi^{1/n}a$$
descends to an endomorphism of $\mathrm{H}^0(\Psi^{1/P^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_P(A/R)/(q-1))$, which we may denote by $V_n$ because it mimics Verschiebung in the sense that $\Psi^n V_n = n\cdot \mathrm{id}$.

For $A$ smooth over $\mathbb{Z}$, we then have
$$\begin{aligned}\mathrm{H}^0(\Psi^{1/P^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_P(A/\mathbb{Z})/(q-1))/(V_p \ : \ p\in P) &\cong A[q^{1/P^\infty}]/([p]_{q^{1/p}} \ : \ p \in P)\\ &\cong A[\zeta_{P^\infty}],\end{aligned}$$
for $\zeta_n$ a primitive $n$th root of unity.

By adjunction, this gives an injective map
$$\mathrm{H}^0(\Psi^{1/P^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_P(A/\mathbb{Z})/(q-1)) \hookrightarrow W^{(P)}A[\zeta_{P^\infty}]$$
of $\Lambda_P$-rings, which becomes an isomorphism on completing $\Psi^{1/P^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}(A/\mathbb{Z})$ with respect to the system $\{([n]_{q^{1/n}})\}_{n\in P^\infty}$ of ideals, where we write $P^\infty$ for the set of integers whose prime factors are all in $P$. This implies that the cokernel is annihilated by all elements of $(q^{1/P^\infty}-1)$, so leads us to consider almost mathematics as in [GR].

3.2. **Almost isomorphisms.** Combined with Lemma 3.3, Remark 3.4 allows us to regard $\mathbf{L}\eta_{(q-1)}\Psi^{1/p^\infty}\mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(A/\mathbb{Z}_p)^{\wedge_p}$ as being almost a $q^{1/p^\infty}$-analogue of $p$-typical de Rham–Witt cohomology. (From now on, we consider only the case $P=\{p\}$.)

The ideal $(q^{1/p^\infty}-1)^{\wedge_{(p,q-1)}} = \ker(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}} \to \mathbb{Z}_p)$ is equal to the $p$-adic completion of its square, since we may write it as the kernel $W^{(p)}(\mathfrak{m})$ of $W^{(p)}(\mathbb{F}_p[q^{1/p^\infty}]^{\wedge_{(q-1)}}) \to W^{(p)}(\mathbb{F}_p)$, for the idempotent maximal ideal $\mathfrak{m} = ((q-1)^{1/p^\infty})^{\wedge_{(q-1)}}$ in $\mathbb{F}_p[q^{1/p^\infty}]^{\wedge_{(q-1)}}$. If we set $h^{1/p^n}$ to be the Teichmüller element
$$[q^{1/p^n}-1] = \lim_{r\to\infty}(q^{1/p^{nr}}-1)^{p^r} \in \mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}},$$
then $W^{(p)}(\mathfrak{m}) = (h^{1/p^\infty})^{\wedge_{(p,h)}}$. Although $W^{(p)}(\mathfrak{m})/p^n$ is not maximal in $\mathbb{Z}[h^{1/p^\infty}]^{\wedge_{(h)}}/p^n$, it is idempotent and flat, so gives a basic setup in the sense of [GR, 2.1.1]. We thus regard the pair $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}, W(p)(\mathfrak{m}))$ as an inverse system of basic setups for almost ring theory.

We then follow the terminology and notation of [GR], studying $p$-adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}})^a$-modules (almost $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$-modules) given by localising at almost isomorphisms, the maps whose kernel and cokernel are $W^{(p)}(\mathfrak{m})$-torsion.

The obvious functor $(-)^a$ from modules to almost modules has a right adjoint $(-)_*$, given by $N_* := \mathrm{Hom}_{\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}}(W^{(p)}(\mathfrak{m}), N)$, the module of almost elements. Since the counit $(M_*)^a \to M$ of the adjunction is an (almost) isomorphism, we may also regard almost modules as a full subcategory of the category of modules, consisting of those $M$ for which the natural map $M \to (M^a)_*$ is an isomorphism. We can define $p$-adically complete $(\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}})^a$-algebras similarly, forming a full subcategory of $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$-algebras.

**Lemma 3.5.** *For any $\mathbb{Z}[\![q-1]\!]$-module $M$, we may recover the $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$-module $(M\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{1/p^\infty}])^{\wedge_p}$ as the module of almost elements of the associated almost $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$-module.*

*Proof.* Since $M\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{1/p^\infty}] = \bigoplus_\alpha M\otimes q^\alpha$ for $\alpha \in p^{-\infty}\mathbb{Z}$ with $0\le\alpha<1$, calculation shows that $(M\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{1/p^\infty}])^{\wedge_p} \to (M\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{1/p^\infty}])^{\wedge_p}_*$ must be an isomorphism. $\square$

3.3. **Perfectoid algebras.** We now relate Scholze's perfectoid algebras to a class of $\Lambda_p$-rings.

**Definition 3.6.** Define a perfectoid $\Lambda_p$-ring to be a flat $p$-adically complete $\Lambda_p$-algebra over $\mathbb{Z}_p$, on which the Adams operation $\Psi^p$ is an isomorphism.

For a perfectoid field $K$ in the sense of [Sch1], there is a tilt $K^\flat$ (a complete perfect field of characteristic $p$). The subring of power-bounded elements is denoted $K^o \subset K$.

**Lemma 3.7.** *Given a perfectoid field $K$, we have equivalences*

$$\begin{array}{c} \textit{perfectoid almost } K^o\textit{-algebras} \\ {\scriptstyle K^{oa}\otimes_{(\mathscr{A}_{\mathrm{inf}}(K^o)^a)}-}\big\uparrow\ \big\downarrow{\scriptstyle \mathscr{A}_{\mathrm{inf}}} \\ \textit{perfectoid almost } \Lambda_p\textit{-rings over } \mathscr{A}_{\mathrm{inf}}(K^o) \\ {\scriptstyle \mathbb{F}_p\otimes_{\mathbb{Z}_p}-}\big\downarrow\ \big\uparrow{\scriptstyle W^{(p)}} \\ \textit{perfectoid almost } K^{\flat o}\textit{-algebras} \end{array}$$

*of categories, where $\mathscr{A}_{\mathrm{inf}}(C) := \varprojlim_{\Psi^p} W^{(p)}(C)$.*

*Proof.* A perfectoid $\Lambda_p$-ring $B$ is a deformation of the $\mathbb{F}_p$-algebra $B/p$. As in [Sch1, Proposition 5.13], a perfect $\mathbb{F}_p$-algebra $C$ has a unique deformation $W^{(p)}C$ over $\mathbb{Z}_p$, to which Frobenius must lift uniquely; this gives the bottom pair of equivalences.

We then observe that since $B := \mathscr{A}_{\mathrm{inf}}(C)$ is a perfectoid $\Lambda_p$-ring for any flat $p$-adically complete $\mathbb{Z}_p$-algebra $C$, we must have $B \cong W^{(p)}(B/p)$. Comparing rank 1 elements then gives a monoid isomorphism $(B/p) \cong \varprojlim_{x\mapsto x^p} C$, from which it follows that

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathscr{A}_{\mathrm{inf}}(C) \cong \varprojlim_{\Phi}(C/p) = C^\flat$$

whenever $C$ is perfectoid. Since tilting gives an equivalence of almost algebras by [Sch1, Theorem 5.2], this completes the proof. ☐

We will only apply Lemma 3.7 to perfectoid almost $\Lambda_p$-rings over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$, in which case it shows that reduction modulo $[p]_{q^{1/p}}$ (resp. $p$) gives an equivalence with perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras (resp. perfectoid $(\mathbb{F}_p[q^{1/p^\infty}]^{\wedge_{(q-1)}})^a$-algebras),

3.4. **Functoriality of $q$-de Rham cohomology.** Since $(\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p))^{\wedge_p}$ is represented by a cosimplicial perfectoid $\Lambda_p$ ring over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$ for any flat $\Lambda_p$-ring $A$ over $\mathbb{Z}_p$, it corresponds under Lemma 3.7 to a cosimplicial perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra, representing the following functor:

**Lemma 3.8.** *For a perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra $C$, and a $\Lambda_p$-ring $A$ over $\mathbb{Z}_p$ with $X = \mathrm{Spec}\, A$, there is a canonical isomorphism*

$$X^q_{\mathrm{strat}}(\varprojlim_{\Psi^p} W^{(p)}(C)_*) \cong \mathrm{Im}\,(\varprojlim_{\Psi^p} X(C_*) \to X(C_*)),$$

*for the ring $C_*$ of almost elements.*

*Proof.* By definition, $X^q_{\mathrm{strat}}(\varprojlim_{\Psi^p} W^{(p)}(C)_*)$ is the image of

$$\mathrm{Hom}_{\Lambda_p}(A, \varprojlim_{\Psi^p} W^{(p)}(C)_*) \to \mathrm{Hom}_{\Lambda_p}(A, (\varprojlim_{\Psi^p} W^{(p)}(C)_*)/(q-1)).$$

Since right adjoints commute with limits, we may rewrite the first term as $\varprojlim_{\Psi^p} \mathrm{Hom}_{\Lambda_p}(A, W^{(p)}(C_*)) = \varprojlim_{\Psi^p} X(C_*)$.

Setting $B := \varprojlim_{\Psi^p} W^{(p)}(C)_*$, observe that because $[p^n]_{q^{1/p^n}}(q^{1/p^n}-1) = (q-1)$, we have $\bigcap_n [p^n]_{q^{1/p^n}} B = (q-1)B$, any element on the left defining an almost element of $(q-1)B$, hence a genuine element since $B = B_*$ is flat. Then note that since the projection map $\theta\colon B \to C_*$ has kernel $([p]_{q^{1/p}})$, the map $\theta \circ \Psi^{p^{n-1}}$ has kernel $([p]_{q^{1/p^n}})$, and so $B \to W^{(p)}C_*$ has kernel $\bigcap_n [p^n]_{q^{1/p^n}} B$. Thus

$$\mathrm{Hom}_{\Lambda_p}(A, (\varprojlim_{\Psi^p} W^{(p)}(C)_*)/(q-1) \hookrightarrow \mathrm{Hom}_{\Lambda_p}(A, W^{(p)}C_*) = X(C_*).$$

□

In fact, the tilting equivalence gives $\varprojlim_{\Psi^p} X(C_*) \cong X(C_*^\flat)$, so the only dependence of $((\Psi^{1/p^\infty}\widehat{\mathrm{qDR}}_p(A/\mathbb{Z}_p))^{\wedge_p})^a$ on the Frobenius lift $\Psi^p$ is in determining the image of $X(C_*^\flat) \to X(C_*)$.

Although this map is not surjective, it is almost so in a precise sense, which we now use to establish independence of $\Psi^p$, showing that, up to faithfully flat descent, $\mathrm{q\hat{D}R}_p(A/\mathbb{Z}_p)^{\wedge_p}/[p]_{q^{1/p}}$ is the best possible perfectoid approximation to $A[\zeta_{p^\infty}]^{\wedge_p}$.

**Definition 3.9.** Given a functor $X$ from $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras to sets and a functor $\mathscr{A}$ from perfectoid $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebras to abelian groups, we write

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}(X, \mathscr{A}) := \mathbf{R}\mathrm{Hom}_{[\mathrm{Pfd}((\mathbb{Z}_p[\zeta_{p^\infty}]^{\wedge_p})^a),\mathrm{Set}]}(X, \mathscr{A}),$$

where $\mathrm{Pfd}(S^a)$ denotes the category of perfectoid almost $S$-algebras, and $[\mathcal{C}, \mathrm{Set}]$ denotes set-valued functors on $\mathcal{C}$. When $X$ is representable by a $(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a$-algebra $C$, we simply denote this by $\mathbf{R}\Gamma_{\mathrm{Pfd}}(C, \mathscr{A})$ — when $C$ is perfectoid, this will just be $\mathscr{A}(C)$.

The following gives a refinement of [BMS, Theorem 1.17], addressing some of the questions in [BMS, Remark 1.11]:

**Theorem 3.10.** *If $R$ is a $p$-adically complete $\Lambda_p$-ring over $\mathbb{Z}_p$, and $A$ a formal $R$-deformation of a smooth ring over $(R/p)$, then the complex*

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge_p}, \mathscr{A}_{\mathrm{inf}})$$

*of $\mathbb{Z}_p[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$-modules is almost quasi-isomorphic to $(\Psi^{1/p^\infty}\mathrm{q\hat{D}R}_p(A/R))^{\wedge_p}$ for any $\Lambda_p$-ring structure on $A$ coming from a framing over $R$ as in Theorem 2.7.*

*Proof.* First observe that $((\Psi^{1/p^\infty}\mathrm{q\hat{D}R}_p(A/R))^{\wedge_p})^a$ is the completion of $\mathrm{q\hat{D}R}_p(A/R)$ with respect to the category of cosimplicial perfectoid almost $\Lambda_p$-rings over $\mathbb{Z}[q^{1/p^\infty}]^{\wedge_{(p,q-1)}}$. Combining the definition of $\widehat{\mathrm{qDR}}_p$ with Lemma 3.7, it then follows that for $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, R$, the complex $(\widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p})_*$ is given by the homotopy limit

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((X^q_{\mathrm{strat}} \times_{Y^q_{\mathrm{strat}}} Y) \circ (\mathscr{A}_{\mathrm{inf}})_*, (\mathscr{A}_{\mathrm{inf}})_*).$$

Writing $X^\infty(C) := \mathrm{Im}\,(\varprojlim_{\Psi^p} X(C_*) \to X(C_*))$, Lemma 3.8 then combines with the description above to give

$$\begin{aligned}(\widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p})_* &\simeq \mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty \times_{Y^\infty} \varprojlim_{\Psi^p} Y, (\mathscr{A}_{\mathrm{inf}})_*),\\ &\simeq \mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty \times_Y \varprojlim_{\Psi^p} Y, (\mathscr{A}_{\mathrm{inf}})_*).\end{aligned}$$

We now introduce a Grothendieck topology on the category $[\mathrm{Pfd}_{(\mathbb{Z}[\zeta_{p^\infty}]^{\wedge_p})^a}, \mathrm{Set}]$ by taking covering morphisms to be those maps $C \to C'$ of perfectoid algebras which are almost faithfully flat modulo $p$. Since $C^\flat = \varprojlim_\Phi (C/p)$, the functor $\mathscr{A}_{\inf}$ satisfies descent with respect to these coverings, so the map

$$\mathbf{R}\Gamma_{\mathrm{Pfd}}((X^\infty \times_Y \varprojlim_{\Psi^p} Y)^\sharp, (\mathscr{A}_{\inf})_*) \to \mathbf{R}\Gamma_{\mathrm{Pfd}}(X^\infty \times_Y \varprojlim_{\Psi^p} Y, (\mathscr{A}_{\inf})_*)$$

is a quasi-isomorphism, where $(-)^\sharp$ denotes sheafification.

In other words, the calculation of $(\widehat{\mathrm{qDR}}_p(A/R)^{\wedge_p})^a$ is not affected if we tweak the definition of $X^\infty$ by taking the image sheaf instead of the image presheaf. We then have

$$(X^\infty)^\sharp(C) = \bigcup_{C \to C'} \mathrm{Im}\,(X(C_*) \times_{X(C'_*)} \varprojlim_{\Psi^p} X(C'_*) \to X(C_*)),$$

where $C \to C'$ runs over all covering morphisms.

Now, $\varprojlim_{\Psi^p} X$ is represented by the perfectoid algebra $(\Psi^{1/p^\infty} A)^{\wedge_p}$, which is isomorphic to $A[x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge_p}$ as in the proof of Lemma 3.3. This allows us to appeal to André's results [And, §2.5] as generalised in [Bha, Theorem 2.3]. For any morphism $f\colon A \to C$, there exists a covering morphism $C \to C_i$ such that $f(x_i)$ has arbitrary $p$-power roots in $C_i$. Setting $C' := C_1 \otimes_C \ldots \otimes_C C_d$, this means that the composite $A \xrightarrow{f} C \to C'$ extends to a map $(\Psi^{1/p^\infty} A)^{\wedge_p} \to C'$, so $f \in (X^\infty)^\sharp(C)$. We have thus shown that $(X^\infty)^\sharp = X$, giving the required equivalence

$$((\Psi^{1/p^\infty} \mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(A/R))^{\wedge_p})_* \simeq \mathbf{R}\Gamma_{\mathrm{Pfd}}(X \times_Y \varprojlim_{\Psi^p} Y, (\mathscr{A}_{\inf})_*).$$

$\square$

**Corollary 3.11.** *If $R$ is a $p$-adically complete $\Lambda_p$-ring over $\mathbb{Z}_p$, and $A$ a formal $R$-deformation of a smooth ring over $(R/p)$, then the $q$-de Rham cohomology complex $(q\text{-}\Omega^\bullet_{A/R,\square} \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge_p}$ is, up to quasi-isomorphism, independent of a choice of co-ordinates $\square$. It is naturally an invariant of the commutative $p$-adically complete $(\Psi^{1/p^\infty} R)[\zeta_{p^\infty}]^{\wedge_p}$-algebra $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge_p}$.*

*Proof.* By Theorem 3.10, we know that the complex $((\Psi^{1/p^\infty} \mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(A/R))^{\wedge_p})_*$ depends only on $(A[\zeta_{p^\infty}] \otimes_R \Psi^{1/p^\infty} R)^{\wedge_p}$. Since

$$\Psi^{1/p^\infty} \mathrm{qDR}_p(A/R) = \Psi^{1/p^\infty} \mathrm{qDR}_p((A \otimes_R \Psi^{1/p^\infty} R)/\Psi^{1/p^\infty} R),$$

Theorem 2.7 combines with Lemmas 3.3 and 3.5 to give

$$(q\text{-}\Omega^\bullet_{A/R,\square} \otimes_{R[q]} (\Psi^{1/p^\infty} R)[q^{1/p^\infty}])^{\wedge_p} \simeq \mathbf{L}\eta_{(q-1)}((\Psi^{1/p^\infty} \mathrm{q}\hat{\mathrm{D}}\mathrm{R}_p(A/R))^{\wedge_p})_*,$$

which completes the proof. $\square$

*Remark* 3.12 (Scholze's conjectures). If we weaken the conjectures of [Sch2] by attaching roots of $q$, then most of them follow from the results above. Taking $R = \mathbb{Z}_p$ in Corollary 3.11 establishes the analogue of [Sch2, Conjecture 3.1] (co-ordinate independence of $q$-de Rham cohomology over $\mathbb{Z}$) via an arithmetic fracture square. Taking more general base rings $R$ in Corollary 3.11 gives an analogue of [Sch2, Conjecture 7.1], further weakened by having to invert all Adams operations on $R$.

The description of Theorem 3.10 is very closely related to the definition of $A\Omega^\bullet$ in [BMS, Definition 1.12], giving an analogue of [Sch2, Conjecture 4.3], and hence the comparison with singular cohomology in [Sch2, Conjecture 3.3]. The operations described in [Sch2, Conjectures 6.1 and 6.2] correspond to the Adams operations on qDR respectively at and away from the residue characteristics. Remark 1.19 provides a category of $q$-connections as described in [Sch2, Conjecture 7.5]; these will correspond via Theorem 1.11 to projective $\mathscr{A}_{\mathrm{inf}}$-modules on the site of integral perfectoid algebras over $A[\zeta_{p^\infty}]^{\wedge_p}$, so are again independent of co-ordinates after base change to $\mathbb{Z}[q^{1/P^\infty}]$.

## References


[And] Y. Andre. La conjecture du facteur direct. arXiv: 1609.00345 [math.AG], 2016.

[Bha] B. Bhatt. On the direct summand conjecture and its derived variant. arXiv:1608.08882[math.AG], 2016.

[BMS] B. Bhatt, M. Morrow, and P. Scholze. Integral $p$-adic Hodge theory. arXiv:1602.03148 [math.NT], 2016.

[Bor] James Borger. The basic geometry of Witt vectors, I: The affine case. *Algebra Number Theory*, 5(2):231–285, 2011.

[GR] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003. arXiv:math/0201175v3 [math.AG].

[Gro] A. Grothendieck. Crystals and the de Rham cohomology of schemes. In *Dix Exposés sur la Cohomologie des Schémas*, pages 306–358. North-Holland, Amsterdam, 1968.

[Joy] André Joyal. $\delta$-anneaux et vecteurs de Witt. *C. R. Math. Rep. Acad. Sci. Canada*, 7(3):177–182, 1985.

[Sch1] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sch2] Peter Scholze. Canonical $q$-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math.*, to appear. arXiv:1606.01796v1[math.AG].

[Sim] Carlos Simpson. Homotopy over the complex numbers and generalized de Rham cohomology. In *Moduli of vector bundles (Sanda, 1994; Kyoto, 1994)*, volume 179 of *Lecture Notes in Pure and Appl. Math.*, pages 229–263. Dekker, New York, 1996.